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Collision orbits of a swinging Atwood's machine

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Résumé. — Nous analysons le mouvement d'un système mécanique non linéaire au voisinage d'une singularité en utilisant une technique géométrique simple due à McGehee. Cette analyse peut se révéler utile dans la détermination de l'existence d'un comportement intégrable ou chaotique.

Abstract. — The motion of a nonlinear mechanical system is analysed in the neighbourhood of a singularity using a simple geometrical technique due to McGehee. The analysis can prove useful in determining the existence of integrable or chaotic behaviour.

1. Introduction.

A statement often made in classical mechanics is « the state of the system is known for all time if its initial state is known » [1]. This comment can be misleading on two counts. First, although deterministic, a Hamiltonian system may still be « stochastic » in the sense that it exhibits sensitive dependence to initial conditions [2]. The term « chaos » has been coined for such deterministic randomness and the study of chaos in Hamiltonian systems is currently an area of intensive research. A nice introduction to the field is the book « Regular and Stochastic Motion » [3].

Second, Hamiltonian systems that arise in applications often have « singularities ». By a singularity of a classical mechanical system we mean a point where the acceleration is undefined. The simplest example of a singularity is the collision of two or more point particles in the Newtonian n-body problem. At a collision the differential equations are no longer defined and, moreover, orbits that pass close to a singularity can behave in an erratic manner. Orbits which are initially close can end up far apart after passing near a singularity. Again, such sensitive dependence on initial conditions is the signature of chaos. Therefore in the case of collision orbits, we see that the state of the system is not known for all time, but only until the time of collision. Thus singularities can be the source of chaotic behaviour.

The question naturally arises as to the possibility of extending collision orbits through a singularity. This is called the « regularization » problem. In this paper we will solve the regularization problem for a simple nonlinear mechanical system, — a Swinging Atwood's Machine (SAM), — using a geometrical technique devised by McGehee [4]. McGehee's idea is to make a change of variables which will remove the singularity from the Hamiltonian system. A good introduction to the technique is given by Devaney [5, 6]. However, the potential for SAM is not of a type previously studied, and consequently we employ a variable transformation different from that used by Devaney. We shall see that with the McGehee technique, orbits which pass near the singularity are simple to understand.

At the other end of the spectrum from chaotic motion is integrable motion. Recently it was shown that SAM is integrable when the mass ratio is three [7]. The first clues suggesting integrability came from the singularity analysis shown in this paper.

Our aim then is to illustrate a new application of the McGehee technique and to suggest that such an analysis may be useful in indicating when a Hamiltonian system with singularities is integrable.

2. SAM equations.

SAM is a simple Atwood's machine in which one of the weights is allowed to swing in a plane (Fig. 1). In polar coordinates the equations of motion and energy are [8] :

$$(1 + \mu)\ddot{r} = r\dot{\theta}^2 + g(\cos(\theta) - \mu) \tag{1a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin(\theta) = 0\,, \tag{1b}$$

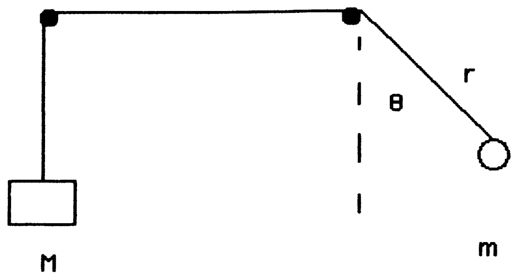

Fig. 1. — SAM : swinging Atwood's machine.

and

$$E = T + V = \tfrac{1}{2}(1 + \mu)\,\dot{r}^2 + \tfrac{1}{2}\,r^2\,\dot{\theta}^2 + g r(\mu - \cos(\theta)) \quad (2)$$

where g is the acceleration due to gravity and $\mu = M/m$ is the mass ratio. The equations are singular and the acceleration is discontinuous at $r = 0$. A certain class of orbits will begin and end at the singularity. If $r(t) \to 0$ as t increases we call the orbit a « collision ». Alternatively, an « ejection » orbit is one in which $r(t) \to 0$ as t decreases. There is no obvious way to continue orbits through the singularity. In the next section we shall see that ejection/collision orbits can be continued (regularized) only for certain values of μ.

3. McGehee transformation.

By making a simple change of variables and rescaling time it is possible to understand the behaviour of orbits which pass close to the singularity. We wish to examine the motion near the singularity as if it were under a microscope and in slow motion. To do this we first transform our second order system to a first order system in the new variables

$$s = \dot{r}\,, \quad (3a)$$

and

$$u = r\dot{\theta}\,, \quad (3b)$$

where s is the radial velocity and u is the tangential velocity. The equations of motion (1) become

$$\dot{r} = s\,, \quad (4a)$$

$$\dot{\theta} = u/r\,, \quad (4b)$$

$$\dot{s} = \frac{1}{1 + \mu}\left[\frac{u^2}{r} + g(\cos(\theta) - \mu)\right], \quad (4c)$$

$$\dot{u} = -\frac{su}{r} - g\sin(\theta)\,. \quad (4d)$$

The system (4) is still singular at $r = 0$, however this singularity can now be removed by choosing a new time scale so that $\mathrm{d}t/\mathrm{d}\tau = r$, i.e.,

$$\tau = \int \frac{\mathrm{d}t}{r(t)}\,. \quad (5)$$

Letting a prime denote differentiation with respect to τ we obtain from (4)

$$r' = sr\,, \quad (6a)$$

$$\theta' = u\,, \quad (6b)$$

$$s' = \frac{1}{1 + \mu}\left[u^2 + gr(\cos(\theta) - \mu)\right], \quad (6c)$$

and

$$u' = -su - gr\sin(\theta)\,. \quad (6d)$$

Equations (6) are no longer Hamiltonian since a noncanonical transformation is employed.

The vector field (or flow) defined by (6) is equivalent to (1) but the following remarks are in order. First, the vector field is now defined at $r = 0$, but orbits that start at $r = 0$ remain there for all time since $r = 0$ implies $r' = 0$. Second, the energy relation

$$E = \tfrac{1}{2}(1 + \mu)\,s^2 + \tfrac{1}{2}\,u^2 + gr(\mu - \cos(\theta)) \quad (7)$$

is still valid even at $r = 0$. The surface generated by (u, s, θ) at $r = 0$ for a fixed value of the energy will be called the « collision surface » and is denoted by Λ. Third, in the case of a collision, the orbit now takes an infinitely long time to reach $r = 0$, and orbits which pass close to a collision now spend a long time near Λ. Most importantly, continuity of solutions with respect to initial conditions implies that the behaviour of near collision orbits can be gleaned from a knowledge of solutions on Λ.

This motivates us to study the flow on Λ. Setting $r = 0$ in (6) yields

$$\theta' = u\,, \quad (8a)$$

$$s' = \frac{u^2}{1 + \mu}\,, \quad (8b)$$

$$u' = -su\,, \quad (8c)$$

along with the energy (from Eq. (7))

$$2E = u^2 + s^2(1 + \mu)\,. \quad (9)$$

The collision manifold is easily seen to be a torus. A cross-section of the torus is shown in figure 2. The cross-section has two fixed points, the lower one is unstable and represents the entrance angle of a collision orbit. The upper rest point is stable and the starting point for an ejection orbit. The full torus and its flow are illustrated in figure 3. We see that there are two circles of fixed points; the lower circle $\left(u = 0,\ s = -\sqrt{\frac{2E}{1 + \mu}}\right)$ is unstable and the upper circle $\left(u = 0, s = +\sqrt{\frac{2E}{1 + \mu}}\right)$ is stable.

Now, we want to know where orbits starting at the lower circle (collisions) go to (ejections). To understand how collision/ejection orbits match up,

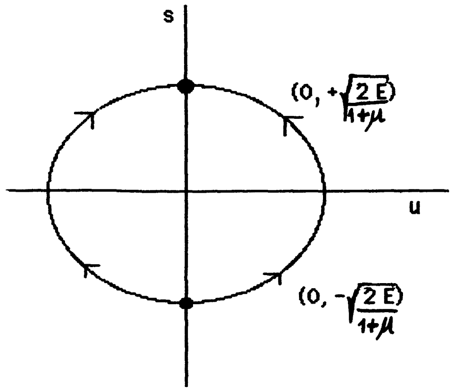

Fig. 2. — Cross-section of the collision manifold. The flow is easily constructed from the equations of motion and energy.

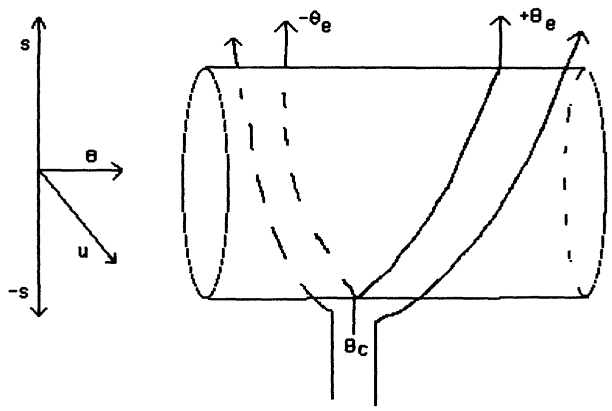

Fig. 3. — The flow on and near the collision manifold. The collision manifold is a torus so $\theta = 0$ and $\theta = 2\pi$ are identified.

equation (8) can be solved exactly by noting

$$\frac{d\theta}{d\tau} = u \tag{10}$$

so $d\tau/d\theta = 1/u$. Hence

$$\frac{du}{d\theta} = -s \tag{11}$$

and

$$\frac{ds}{d\theta} = \frac{1}{1+\mu} u, \tag{12}$$

i.e.,

$$\frac{d^2u}{d\theta^2} = \frac{-1}{1+\mu} u, \tag{13}$$

which has a solution of the form

$$u(\theta) = A \cos\left(\frac{1}{\sqrt{1+\mu}}\theta - \delta\right), \tag{14}$$

where A and δ are contants. Letting θ_c be the « entering angle » and θ_e be the « exiting angle », then $u(\theta_c) = u(\theta_e) = 0$ since the tangential velocity u must be zero to get onto and off of the collision manifold. This implies

$$\cos\left(\frac{1}{\sqrt{1+\mu}}\theta_c - \delta\right) = 0 = \cos\left(\frac{1}{\sqrt{1+\mu}}\theta_e - \delta\right). \tag{15}$$

A little algebra and noting that θ is 2π periodic gives

$$\theta_e = \theta_c \pm \sqrt{1+\mu}\,\pi. \tag{16}$$

Hence, in general, there are two possible exiting angles for a given collision angle. To understand how the sign ambiguity arises consider an orbit which passes close to collision as depicted in figure 4. If the orbit comes in a little above the singularity it will wrap around by $+\sqrt{1+\mu}$ before exiting; however, if it comes in a little below the singularity it will wrap around by $-\sqrt{1+\mu}$ before exiting.

In general, singularities can cause instability since arbitrarily close trajectories can end up far apart after passing near the neighbourhood of a singularity. The exiting angle is not well defined (and the problem is « nonregularizeable ») except in the special case where $\theta_e = \theta_c \pm 2\pi$ which implies

$$\mu = 4n^2 - 1, \quad n \in \mathbb{Z}^+. \tag{17}$$

When $\mu = 3, 15, 35...$, the sensible way to regularize the flow is by the rule

$$\theta_e = \theta_c + 2\pi n, \tag{18}$$

i.e., let the collision angle equal the exit angle. For all other values of μ there appears to be no natural way to regularize the flow.

We have investigated the case $\theta_e = \theta_c + N\pi$ for N odd. However, the numerical evidence suggests that these cases are not integrable.

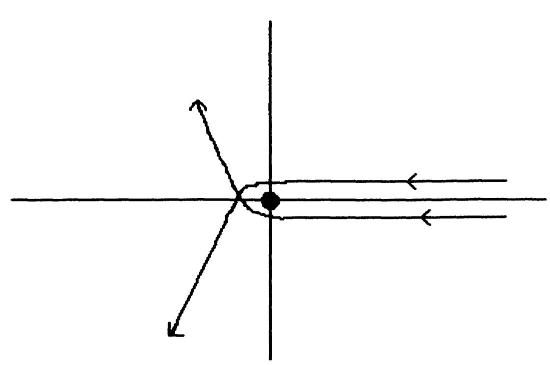

Fig. 4. — Near collision orbits of SAM. The (r, θ) plane is depicted around $r = 0$. The singularity quickly seperates nearby trajectories.

4. Conclusion.

The type of singularity analysis illustrated in the previous section can be used to indicate when a Hamiltonian system with singularities will, or will not, exhibit chaotic motion. To wit, if the flow is not regularizeable we expect the singularity to serve as a source of instability for the system. Alternatively, if the flow is regularizeable, this indicates that the system may be integrable since regularizeability is a necessary (but not sufficient) condition for integrability. In fact, SAM is integrable for $\mu = 3$ and there is numerical evidence to indicate SAM is integrable for the other values of μ specified by equation (17) [9].

The McGehee technique also points out the role noncanonical transformations can play in classical mechanics. Such transformations are rarely mentioned in classical mechanics courses because of the emphasis placed on canonical transformations due to their greater theoretical value.

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