REED COLLEGE

PHYSICS SEMINAR SCHEDULE

ALL SEMINARS ARE HELD AT 4:10 ON WEDNESDAY IN ROOM P123 UNLESS OTHERWISE INDICATED.

	February 3	MAYHEW, TUFILLARO*
SPECIAL	February 4, Thursday, VLH	RICHARD SRAMEK, NRAO, "RADIO SUPERNOVAE"
	February 10	FITZSIMMONS, OWEN
	February 17	BENACQUISTA, WIENER
SPECIAL	February 18, Thursday	PETER OLDS, Reed Chemistry, "ATMOSPHERIC TRACE GASES"**
	February 24	BONAR, MCNULTY
	March 3	LITT, MCGEHEE
SPECIAL	March 4, Thursday, VLH, 8:00 P.M.	STEVEN MORSE, INTEL, "HISTORY OF THE INTEL MICROPROCESSOR"
	March 10	HENLEY, WEDELL
	SPRING RECESS	
	March 24	MCCORMICK, WRIGHT
	March 31	HELEN QUINN, SLAC, A.A. Knowlton Visiting Lecturer: Technical seminar; topic to be announced.
SPECIAL	March 31, 8:00 P.M. VLH	HELEN QUINN, General lecture; topic to be announced.
	April 7	ROBINSON, RUF
SPECIAL	April 13, Tuesday, 8:00 P.M., VLH	WOODRUFF SULLIVAN III, University of Washington, Harlow Shapley Visiting Lecturer: "THE SEARCH FOR EXTRATERRESTRIAL INTELLIGENCE—A PLAN FOR ACTION"
	April 14, VLH	WOODRUFF SULLIVAN III, "THE AGE AND SIZE OF THE UNIVERSE"
	April 21	PAGLIN, SUTTON
	April 28	CALVERT, GRONKE

**Another in the series "Summer Research Activities—1981"

*Unless otherwise indicated all speakers are Reed physics seniors.

Physics Seminar   Reed College   3/2/82

Title: Smiling Swingers or 4012 Physics.

Goodafternoon. I intend to describe a system from my note-book of Classical Mechs. problems which on the surface looks simple, like a toy at best, but whose behavior is astonishingly complex and presents facets of more than academic luster. It is a problem concerning the orbits or trajectories of a particle in two-degrees of freedom under a very particular non-central force; to wit, the problem of two-pendula. Now, the difficulty of this problem should really suprise noone; for the general problem of motion of a classical particle in two degrees of freedom is no where near being solved. As the russian mathmatician V. I. ARNOLD comments in his 1980 book on Classical Mechanics:

?? unclear

"Analyzing a general potential system with two degrees of freedom is beyond the capability of modern science."

an atwoods machine in which one of the masses is allowed to swing

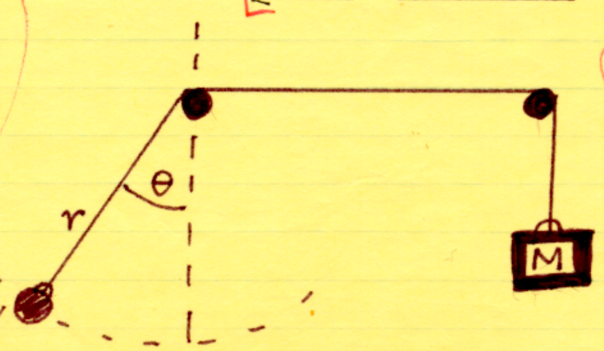
The problem I wish to share with you, ~~the two pendula problem (t.p.p.)~~; ~~it~~ consists of describing the motions of the following machine: (go to board; do not pass go, do not collect $200)

I think this name is misleading, since one of them is NOT a pendulum

I think a cleaner term would be "Atwoods machine in which one of the masses is allowed to swing".

Two Pendula Machine

Must you be so explicit at this point? They won't catch all this, & it does not apply to teardrop anyways.



Given: at time $t=0$

$r(0)=\ell$, $\theta(0)=\theta_0$, m & M

$\dot{r}(0)=\dot{r}_0$, $\dot{\theta}(0)=\dot{\theta}_0$

Find: What happens?

$\theta(t)$, $r(t)$, $t>0$

Fig. 1

The machine is constructed from two masses, m & M, little 'm' and capital 'M' which are connected by a massless string and supported by two frictionless pullies; and all this takes place in a vacum (in more ways than one), all very realistic and in accordance with the assumptions made in an introductory physics course. The bob, little m, is free to swing, like a simple pendulum, in the plane. the block, capitial M, is confined to move in the vertical direction only, up and down, in one dimension. We futher assume conservation of energy, there is no ~~energy loose due to motion of~~ friction the pulley. ~~or bobs~~. Lastly, we choose to describe the problem in polar coordinates, where $\theta \neq 0$ is measured from a plum line droped from the pulley.

$\theta > 90°$

Now you may look what happens when

At this point we will quickly argue that the machine works when θ is greater than $90°$; i.e., that there is always positive tension in the string: (go to board)

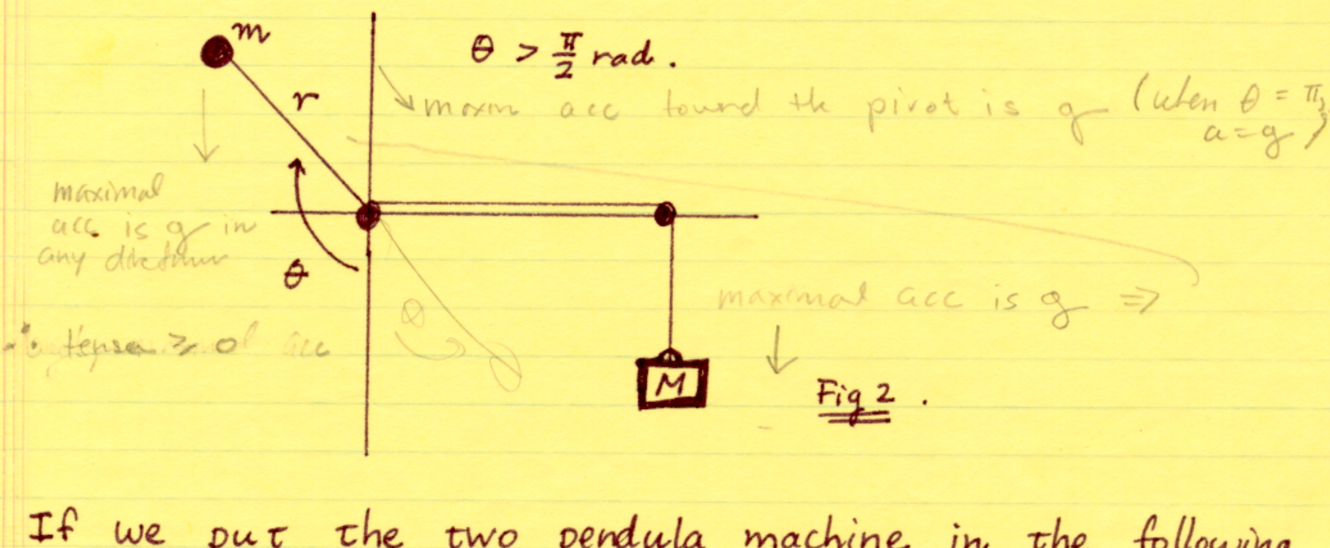


Fig 2.

If we put the two pendula machine in the following configuration and let-go, what happens? Well, unfortunaley, only the computer knows for sure, (pause, hopefully); but seriously, the string bellow little 'm' will not 'collapse'. We recognize, along with Galileo, the constant acceleration of all free falling bodies. Imagine that the string is no longer in the picture. Then m and M are free falling bodies with the same acceleration. When we place the string back into the picture, it is clear that the tension in the string is positive; it is zero when 'm' is ~~always upside down and~~ directly above the pulley.

I'd kill this –

Before we continue, I would like to thank both Prof. Griffiths & Prof. Crandall for generously extending their time and ideas about ~~this~~ PN problem. I am really very ~~grateful to~~ beholden to both.

The first step ~~in~~ to generally discover what happens is to obtain the equations of motion. The standard operating procedure in these matters is to write down the energy equations and then to get the equations of motions by means of the Euler Lagrange's equations. At this point one is usually in a position to give the equations a 'Newtowian interpretation' so that in the future, one can instantly derive the equations of motion by Newtowian principles.

Confusing – wait until it happens.

To this end, we note that the ~~one~~ kinetic energy for the system is:

(go to board)

$$T = \underbrace{\tfrac{1}{2} m \dot{r}^2}_{\frac{1}{2} m v_r^2} + \underbrace{\tfrac{1}{2} m (r^2 \dot{\theta}^2)}_{\frac{1}{2} m v_\theta^2} + \underbrace{\tfrac{1}{2} M \dot{r}^2}_{\frac{1}{2} M v_r^2}$$

$$= \underbrace{\tfrac{1}{2} m (\dot{r}^2 + (r\dot{\theta})^2)}_{\text{just like simple pendulum but } \dot{r} \neq 0.} + \tfrac{1}{2} M \dot{r}^2$$

And the potential energy is similar to that of an atwoods machine with the addition of a $\cos\theta$ term in order to compicate for the swinging side:

$$\boxed{V = -mgr\cos\theta + Mgr + \underbrace{C}_{\text{dependens on where we choose to zero the potential.}}}$$

avoid this; insulting form: It's ("you really oughta know this, but since I suspect you don't, I'm gonna tell you" is what it means)

~~Recall that~~ the Lagragian is ~~defined to be~~ the DIFFERENCE between the kentic ~~potential~~ and potential ~~kentic~~ energies:

$$\boxed{\begin{aligned}\mathcal{L} &= T - V \\ &= \tfrac{1}{2}M\dot{r}^2 + \tfrac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) - Mgr + mgr\cos\theta - C\end{aligned}}$$

Use $\mathcal{L}$ to calculate the equations of motion via Lagrangers celebrated equation:

$$\boxed{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) = \frac{\partial\mathcal{L}}{\partial q_i}}$$

where $q_1 = r$

$q_2 = \theta$

So ~~we proceed with the~~ ~~w~~ if we succed in calculating the partial derivatives correctly we discover

r-eq) $(m+M)\ddot{r} = \underbrace{mr\dot{\theta}^2}_{\text{centripedal force}} + \underbrace{mg\cos\theta - Mg}_{\text{gravitatn force}}$

θ-eq) $\underbrace{\frac{d}{dt}(mr^2\dot{\theta}) = -mgr\sin\theta}$

should be familar from the simple pendulum problem except $\dot{r} \neq 0$.

Newtonian interpretation

$F_r = \sum_i m_i a_r = (m+M)\ddot{r}$

"time rate of change of angular momentum is equal to the applied torque" (angular equation)

As a notational convience we define

$$\alpha = M/m$$

and rewrite eq's of motion as

r-eq) $(1+\alpha)\ddot{r} = r\dot{\theta}^2 + g(\cos\theta - \alpha)$

θ-eq) $\frac{d}{dt}(r^2\dot{\theta}) = -gr\sin\theta$ or,

$$\ddot{\theta} + \frac{2\dot{r}}{r}\dot{\theta} + \frac{1}{r}g\sin\theta = 0$$

the physics

My thesis, is, in a sense, all on the board in this last box. It amounts to an analysis of these equations. On first inspection, these equations look horrible: the trigometric terms and the squared terms make them 'horribly non-linear' (where horriblely is used in the technical sense). But it is exactly the 'mystique' of non-lineraity which becons me to study ther properties. To my mind, they are not horrible, but a rather lovely set of second-order-coupled-non linear ordinary-differential-equations. I want to break these Coupled - Ordinary - Differential - Equations.

One really can't hope hope to find many, non trivial if any, exact solutions to these equations (there is a known ~~two~~ one, but it is for negative mass, alas). ~~the probability of finding such solutions is something like lebeque measue zero.~~

How about Atwood's solutions with $\theta = 0$? That's exact and positive masses

In order then to develop some feel for the system and "soften up the terrority" so to speak, ~~we~~ I began ~~our~~ my ~~studies~~ venture with extensive numerical integration studies which ~~we~~ I latter used as a yard stick to check ~~our~~ my results.

What do you think you are, a King?

Show $\boxed{\text{SET 1}}$ pictures:

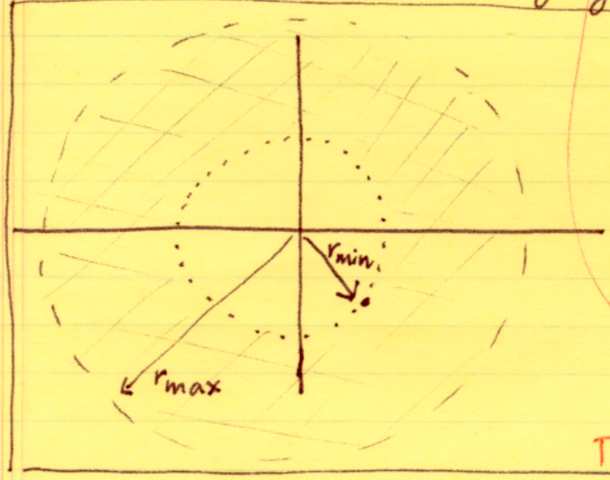
Commentary:

1) Describe initial conditions.
2) Mention each picture shows a mass increase of about 0.1.
3) For $M \leq 1$, it appears that $r(t)$ presents runaway solutions, it is unbounded and never returns. (We will prove this)
4) The motion appears erogotic, periodic, bounded, stabe (recurrent) etc. random

These pictures suggest we look into the following problems and introduce ~~the~~ some terminology.

Observe that this is a hierarchy:
periodic ⇒ stable ⇒ bounded.

We would quickly like to presensent a simple and tentative scheme by which to classify the motions of the swinging bob in the plane.



bounded: $0 \le r(t) \le r_{max} < \infty$

stable:

$0 < r_{min} \le r(t) \le r_{max} < \infty$

periodic: ("commensurable")

$r(t+\tau_1) = r(t)$ and $\theta(t+\tau_2) = \theta(t)$

where $\tau_2 = n\tau_1$ for some rational

Then $\theta(t+\tau_3) = \theta(t)$; $r(t+\tau_3) = r(t)$
where τ_3 is the least common multiple

If the bob always remains within some circle ~~with~~ of radius r_{max}, we shall say the motion is bounded; i.e., if there exists a positive upper bound for $r(t)$.

If there further exists an r_{min} ~~such~~, ~~that~~ i.e, r has a non-zero lower bond, then we will call the motion stable.

Lastly, there appear to exist exactly periodic solutions.

Why not use this as simpler definition of periodic?

The door now opens to a whole slew of problems. Some of the more salient ones are:

① Under what conditions do periodic orbits ~~and~~ occur ~~exist~~ ~~&~~ can we find aproximate or exact solutions for these trajectories → Unclear. FOR WHAT MASSES + STARTING CONDITIONS.

② Are the motions around ~~around~~ periodic orbits stable (Poincare orbital stability) in the sense that a ~~solutions~~ near a periodic solution always stays close to it: is the motion reccurent in ~~bird~~ Birkhoffs sense. → starting conditions + masses

I wouldn't drop names unless you propose to define these terms more precisely

③ Can we discover a classification scheme for the periodic orbits; or write some exact analytic statement about under what conditions, bounded, stable and periodic motion will occur.

and so on.

I would like to show you a second set of trajectories, this time of orbits which are suspected to be periodic on numerical grounds and then to ~~show~~ present some of the more outstanding results which have been obtained in trying to solve these, and related questions.

Show: $\boxed{\text{SET 2}}$

Commentary:

1) There appears to be a 'spectrum' of periodic solutions which are even or odd in the sense that rotation by 180° is even and 360° is odd.
symmetry with y-axis.

As a next step we examine a couple of limiting cases. We will look at

Limiting Cases:

1) Atwoods ~~aproximation~~: $\theta(t)=0, \forall t$.

2) Simple pendula ~~aproximation~~: $\dot{r}=0$, $\theta \ll 1$ $\forall t$.

For the atwoods aproximation we need only look at the r-eq to see that the name is aptly chosen:

$$(1+\alpha)\ddot{r} = \cancelto{0}{r\dot{\theta}^2} + g(\cancelto{1}{\cos\theta} - \alpha)$$

$$= g(1-\alpha) \Rightarrow$$

$$\ddot{r} = g\frac{(1-\alpha)}{(1+\alpha)}$$

which is the acceleration of an atwoods machine with solution

$$r(t) = l + vt + \frac{1}{2}g\left(\frac{1-\alpha}{1+\alpha}\right)t^2 . \quad /// \quad v = \dot{r}_0$$

The simple pendulum aproximation turns out to be some what less trivial, the θ-eq becomes

$$\ddot{\theta} = -\frac{g}{l}\sin\theta \doteq -\frac{g}{l}\theta \qquad (\theta \ll 1).$$

The solution is a simple harmonic oscillator

$$\theta(t) = \theta_0 \cos\omega t \quad , \quad \omega = \sqrt{g/l}$$

Pluging this into the (r-eq) and examining the time average leads to a very intersting conclusion.

$$\langle \underbrace{(1+\alpha)\ddot{r}}_{0} \rangle = \langle r\dot{\theta}^2 \rangle + g\langle \cos\theta \rangle - g\alpha$$, recall $r = l \Rightarrow \dot{r} = \ddot{r}$

so

$$g\alpha = \langle r\dot{\theta}^2 \rangle + g\langle \cos\theta \rangle$$ now

$\theta = \theta_0 \cos \omega t$

$\dot{\theta} = -\theta_0 \omega \sin \omega t$

$\ddot{\theta} = -\theta_0 \omega^2 \cos \omega t$ so

$\cos\theta \approx 1 - \frac{\theta^2}{2} + \dots$

$$g\alpha = \langle l\,\theta_0^2 \omega^2 \sin^2 \omega t \rangle + g\langle \cos(\theta_0 \cos \omega t) \rangle \qquad \omega^2 = g/l$$

$$\cancel{g}\alpha = \cancel{g}\langle \theta_0^2 \sin^2 \omega t \rangle + \langle (1 - \frac{\theta_0^2 \cos^2 \omega t}{2} \rangle$$

$$\frac{1}{2}\theta_0^2 \quad + 1 \quad - \frac{1}{4}\theta_0^2$$

~~$\alpha \approx$~~

So $$\boxed{\alpha = 1 + \frac{\theta_0^2}{4}}$$

Smile

STABILITY FORMULA.

That is, if we can get a solution which resembles a simple pendulum for small angles then we must have the above relation holding for α & θ_0. Notice that this relation is independ of g and l [this really has to be the case since both g and l scale in the original equations of motion, a non-trivial observation].

($\alpha = f(\theta) \Rightarrow$ periodic solu.)

Such a relation we call a 'stablity formula'. This formula agree well with numerical results for small angles ($30° <$). Others have been constructed by numerical methods for different types of periodic solution.

$$T = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 \qquad V = gr(M - m\cos\theta)$$

$$L = T - V$$

$$\boxed{L = \frac{1}{2}(M+m)\dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + gr(m\cos\theta - M)}$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}} \qquad q_1 = r \qquad q_2 = \theta$$

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 + g(m\cos\theta - M)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{d}{dt}\Big((m+M)\dot{r}\Big) = (m+M)\ddot{r}$$

$$\boxed{(m+M)\ddot{r} = m r \dot{\theta}^2 + g(m\cos\theta - M)}$$

$$\boxed{F_r = F_c + F_G}$$

Radial equation

$$\frac{\partial L}{\partial \theta} = -grm\sin\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}(\cancel{m} r^2 \dot{\theta}) = -gr\cancel{m}\sin\theta$$

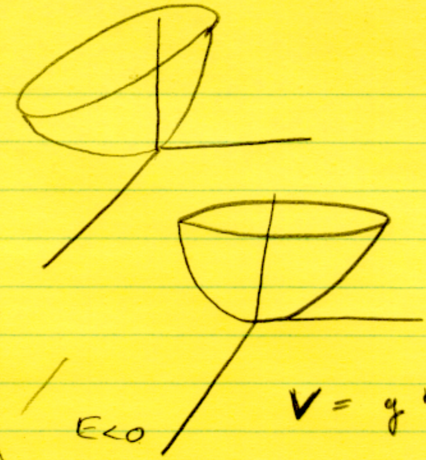
$$\boxed{\frac{d}{dt}(L) = \tau}$$

$$2\cancel{r}\dot{r}\dot{\theta} + \cancel{r}r\ddot{\theta} = -g\cancel{r}\sin\theta$$

$$\boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} = -g\sin\theta}$$

angular equation

Boundedness $\mu > 1$

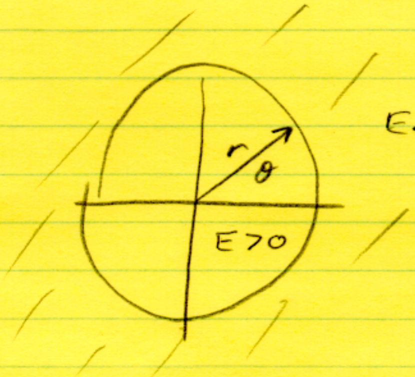


$V = g\,r(\mu - \cos\theta)$

$E = T + V$

$=$

$=$



$$E = \underbrace{\tfrac{1}{2}(1+\mu)\dot{r}^2 + r^2\dot{\theta}^2}_{\text{post definite}} + gr(\mu - \cos\theta)$$

$\geq gr(\mu - 1)$

$\boxed{E \geq gr(\mu - \cos\theta)}$ $\rightarrow$ Alway true $\longrightarrow$ Now if $\mu > 1$ then $(\mu - \cos\theta) > 0$ $\neq 0$

$$\boxed{\frac{E}{g(\mu - \cos\theta)} \geq r} \longleftarrow$$

~~Converse is also true~~

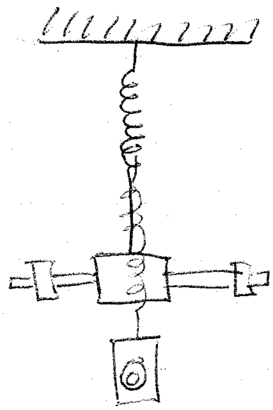
The $\Longleftarrow$ also true but a little more tricky to prove.

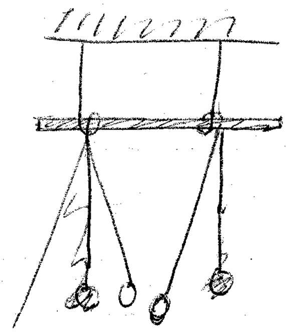
Booker, H.G. A Vector Approach to Oscillations

Magnus, K. Vibrations, Blackie, London 1965

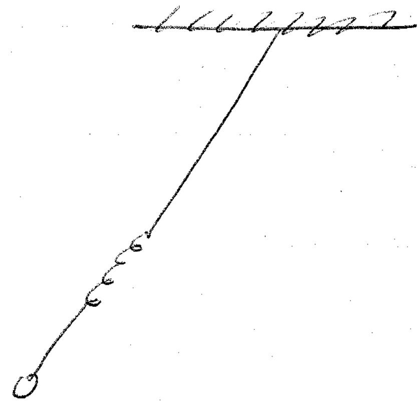
Temple, G. and Bickley, W.G. Rayleigh's Principle. Dover, New York 1956.

Coupled oscillators





Problem 5-1
French



Wilbefore Pendulum
pg 128 French.